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LETTER TO THE EDITOR

On the range of validity of the $6-\varepsilon$ expansion for percolation

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Abstract. In $6-\varepsilon$ dimensions the critical properties of percolation can be computed starting from a φ^3 theory having the same symmetry of the one-state Potts model. In this paper we argue that this fixed point must be unstable with respect to a φ^4 interaction in two dimensions; therefore a φ^4 perturbation is relevant. This suggestion is confirmed by an explicit two-loops computation.

In the framework of the renormalisation group approach to critical phenomena, the fixed point plays a central role (Amit 1978, Brézin *et al* 1976). Normally the structure of the fixed points can be found using the expansion in powers of $\varepsilon = D_c - D$ (for $D > D_c$ only gaussian fixed points are present and the critical exponents are trivial).

Also for ε not small it is generally believed that the results of the ε expansion are qualitatively correct. This seems not to be the case for two-dimensional percolation.

Percolation can be considered as the limit $q \to 1$ of the q-state Potts model (Kasteleyn and Fortuin 1969, 1972); the corresponding field-theoretical Hamiltonian (Wallace and Zia 1975) is

$$H = \int \mathrm{d}^D x [\tfrac{1}{2}(\partial_\mu \varphi_i)^2 + \tfrac{1}{2}\mu^2 \varphi_i^2 + (1/3!) g \varphi_i \varphi_j \varphi_k Q_{ijk} + \mathrm{O}(\varphi^4)], \tag{1}$$

Q_{ijk} being a tensorial coupling (it will be defined in equation (3)); the index i ranges from 1 to N. However if one uses the Hamiltonian (1) to compute (or in the $D = 6-\varepsilon$ expansion or in the fixed-dimension loop expansion) the exponent η, one finds a stable negative result ($\eta = -0.2 \div -0.3$).

This is at variance with the positivity of the index $\beta = (D-2+\eta)\nu/2$. The failure of the renormalisation group leads us to suggest that the interaction becomes unstable with respect to a φ^4 interaction when the dimension D of the space is smaller than $\tilde{D}$ ($\tilde{D} > 2$).

The aim of this note is to compute the anomalous dimension of the φ^4 operator using the loop expansion at fixed dimensions (Parisi 1980). Although the primary target of this paper is percolation, we present the results for the q-component Potts model.

Now let us consider the Hamiltonian (1) to which we shall add a piece, a term whose perturbation we want to analyse, of the form

$$(1/4!) \int \mathrm{d}^D x \ (\lambda_0 S_{ijkl} + \alpha_0 F_{ijkl}) \varphi_i \varphi_j \varphi_k \varphi_l \tag{2}$$

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where, for the Potts model, the tensorial couplings S, F, Q (defined in equation (1)) are (Wallace and Zia 1975)

$$Q_{ijk}=\sum_\alpha l_i^\alpha l_j^\alpha l_k^\alpha, \qquad F_{ijkl}=\sum_\alpha l_i^\alpha l_j^\alpha l_k^\alpha l_l^\alpha, \qquad S_{ijkl}=\tfrac{1}{3}(\delta_{ij}\delta_{kl}+\text{permutations}), \tag{3}$$

where the l_i^α obey

$$\sum_\alpha l_i^\alpha=0, \qquad \sum_\alpha l_i^\alpha l_j^\alpha=(N+1)\delta_{ij}, \qquad \sum_i l_i^\alpha l_i^\beta=(N+1)\delta^{\alpha\beta}-1, \tag{4}$$

and where the Greek indices run from 1 to $N-1$ while the Latin indices run from 1 to N.

Naive dimensional power counting will tell us that for $D>4$ the terms in (2) have negative dimension (the coupling constants λ_0, α_0 have naive dimension $4-D$); is this enough to conclude that the insertions of φ^4 operators are less singular than the φ^3 term and cannot perturb it? Certainly not! We must know the dimension of the φ^4 operator at the non-trivial fixed point, not at the gaussian fixed point.

In principle the dimension of the φ^4 operator can be computed using the Callan–Symanzik equation for the insertion of a φ^4 operator. Unfortunately things are not so simple because of operator mixing. In $6-\varepsilon$ dimensions there are eight operators of naive dimension eight. They are:

$$\begin{aligned}
&O_1(x)=(1/4!)S\varphi^4(x), \qquad O_2(x)=(1/4!)F\varphi^4(x),\\
&O_3(x)=-(1/3!)(m^2)^{-\varepsilon/4}Q_{ijk}\varphi_i(x)\varphi_j(x)\Box\varphi_k(x),\\
&O_4(x)=\tfrac{1}{2}(m^2)^{-\varepsilon/2}\varphi_i(x)\Box^2\varphi_i(x),\\
&O_5(x)=\tfrac{1}{2}(m^2)^{-\varepsilon/2}[\Box\varphi(x)]^2,\\
&O_6(x)=(1/3!)(m^2)^{-\varepsilon/4}\Box[Q_{ijk}\varphi_i(x)\varphi_j(x)\varphi_k(x)],\\
&O_7(x)=\tfrac{1}{2}(m^2)^{-\varepsilon/2}\Box[\varphi_i(x)\Box\varphi_i(x)],\\
&O_8(x)=\tfrac{1}{2}(m^2)^{-\varepsilon/2}\Box^2[\varphi^2(x)],
\end{aligned} \tag{5}$$

where we have inserted the m^2 term to reduce the O_i operators to the same naive dimension. These operators mix among themselves in $6-\varepsilon$ dimensions at the first order in ε. However, in two dimensions these operators will have very different naive dimensions and there is no reason to restrict the choice to this set (e.g. we could also include a φ^5 operator). Note that operators of different naive dimensions mix because the coupling constant is not dimensionless as in six dimensions. After some reflection we have decided to consider only the two φ^4 operators, this being the only sensible approximation.

If we compare the results for the one-loop expansion obtained with (Amit *et al* 1977) and without mixing (see table 1) we see that the results of this approximation are rather satisfactory.

Amit *et al* (1977) considered, at the order of one loop, all the eight operators of formula (5). Doing the computation, they were able to combine these operators in such a way as to obtain a 5×5 matrix from which one could extract the anomalous dimensions of the O_i.

This matrix was reduced again by means of the equation of motion for the fields φ to its final form of a 2×2 matrix. The values of the anomalous dimensions which we have given in table 1 are just the eigenvalues of this matrix. In the fixed-dimension formalism one works in the massive theory and the critical point is reached when the renormalised dimensionless coupling constant g is equal to g_c ($\beta(g_c)=0$). One then defines the renormalised $\varphi_S^{(R)}$, $\varphi_F^{(R)}$ operators which are proportional to the bare ones $\varphi_i^{(R)}=z_{ik}\varphi_k$

Table 1. We compare the anomalous dimension of the φ^4 operator with (λ_w) and without (λ_{wo}) operator mixing. N is the number of components of the field φ_i.
A_1: $(6)\Gamma(D/2)\Gamma[(8-D)/2]/4$, A_2: $(6)A/2$, A_3: $(12)A/2$, A_4: $(8)A/2$, A_5: $(9)B/6$, A_6: $(12)B/6$, A_7: $(3)C$, A_8: $(4)C$.

N	λ_w^1	λ_w^2	λ_{wo}^1	λ_{wo}^2
0	-0.8ε	-3.7ε	-0.9ε	-4.2ε
1	0	-4.8ε	0	-11ε
2	-6.8ε	-12.8ε	-10.3ε	-15.4ε

where $i = F, S$. If one considers the vertex of these renormalised operators with four renormalised φ fields, the z are fixed by the condition

$$\Gamma^{(4)F}_{(\mathrm{R})ijkl} = F_{ijkl}, \qquad \Gamma^{(4)S}_{(\mathrm{R})ijkl} = S_{ijkl}. \tag{6}$$

A Callan–Symanzik equation for the vertex can be easily derived:

$$\begin{aligned}[p^2\partial/\partial p^2 + \beta(g)\partial/\partial g + 2(D-2) + \tfrac{1}{2}N\gamma(g)\delta_{ik} + \gamma_{ik}]\Gamma^{(N)}_{(\mathrm{R})k}(g, m, p) \\ = (1-\gamma(g))\Delta\Gamma^{(N)}_{i(\mathrm{R})}(m, g, p)\end{aligned} \tag{7}$$

where m, g are the renormalised mass and coupling constant, R stands for renormalised, Δ is the operator $m^2\partial/\partial m^2$, $\Gamma^{(N)}_{k(\mathrm{R})}$ is the usual vertex function but with the insertion of an operator φ^4_k, and γ_{ik} (at zero external momentum) is the matrix which will give us the anomalous dimensions of the operators φ^4

In figure 1 we show the Feynman graphs which enter $\Delta\Gamma^{(4)}_{(\mathrm{R})k}$, while in table 2 we illustrate their numerical values and their multiplicities. To make the picture complete we now give also the tensorial couplings for the diagrams of figure 1:

$$\begin{aligned}
(QQS)_{A_1} &= \tfrac{1}{3}(N+1)^2(N-3)S + \tfrac{2}{3}(N+1)F, \\
(QQF)_{A_1} &= (N+1)^3S + (N+1)^2(N-2)F, \\
(QQQQS)_{A_2} &= \tfrac{1}{3}(N+1)^4(N-3)(N-1)S + \tfrac{2}{3}(N+1)^3(N-1)F, \\
(QQQQF)_{A_2} &= (N+1)^5(N-1)S + (N+1)^4(N-1)(N-2)F, \\
(QQQQS)_{A_3} &= \tfrac{1}{3}(N+1)^4(N-3)(N-2)S + \tfrac{2}{3}(N+1)^3(N-2)F, \\
(QQQQF)_{A_3} &= (N+1)^5(N-2)S + (N+1)^4(N-2)^2F, \\
(QQQQS)_{A_4} &= -\tfrac{1}{3}[2(N+1)^5 - 7(N+1)^4]S + [(N+1)^4 - 3(N+1)^3]F, \\
(QQQQF)_{A_4} &= [(N+1)^6 - 4(N+1)^5]S + [(N+1)^6 - 6(N+1)^5 + 10(N+1)^4]F, \\
(QQQQS)_{A_5} &= (QQQQS)_{A_2}, \\
(QQQQF)_{A_5} &= (QQQQF)_{A_2}, \\
(QQQQS)_{A_6} &= [\tfrac{4}{3}(N+1)^4]S + \tfrac{2}{3}[(N+1)^4 - 4(N+1)^3]F, \\
(QQQQF)_{A_6} &= [2(N+1)^6 - 4(N+1)^5]S + [(N+1)^6 - 6(N+1)^5 + 8(N+1)^4]F, \\
(QQQQS)_{A_7} &= \tfrac{1}{3}[(N+1)^6 - 5(N+1)^5 + 10(N+1)^4]S + \tfrac{2}{3}[(N+1)^4 - 4(N+1)^3]F, \\
(QQQQF)_{A_7} &= [(N+1)^6 - 4(N+1)^5]S + [(N+1)^6 - 6(N+1)^5 + 10(N+1)^4]F, \\
(QQQQS)_{A_8} &= -[(N+1)^5 - 3(N+1)^4]S + [(N+1)^4 - 3(N+1)^3]F, \\
(QQQQF)_{A_8} &= -3(N+1)^5S + [(N+1)^6 - 6(N+1)^5 + 11(N+1)^4]F.
\end{aligned} \tag{8}$$

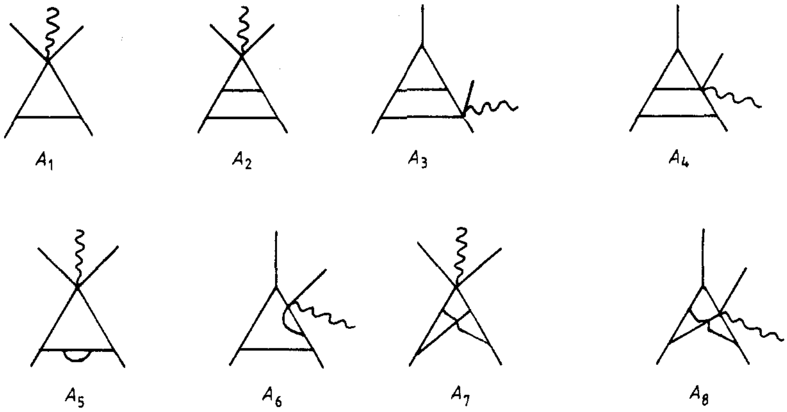

Figure 1. Feynman graphs involved in our computation.

Table 2. Numerical values and multiplicities (in brackets) of the diagrams of figure 1.

D	A	B	C
5	0.145	−0.160	0.2331
4	0.083	−0.063	0.1147
3	0.047	−0.024	0.0582
2	0.027	−0.008	0.0301

Table 3. Anomalous dimension of the φ^4 operator for various D.

D	λ_1	λ_2
5	−7.33	−8.74
4	−5.49	−8.64
3	−3.63	−6.75
2	+1.23	−1.69

These tensorial couplings have been checked by doing the computation first by hand and then using an algebraic computer program (SCHOONSCHIP). At the end we have checked that for the Ising model ($N=1$) one of the two eigenvalues was zero as predicted by the theory. The results in the percolation limit $N\to 0$ (Kasteleyn and Fortuin 1969, 1972) for various values of the dimension are shown in table 3. These values are obtained by diagonalising the matrix

$$\begin{vmatrix} \gamma_{SS}-\lambda & \gamma_{SF} \\ \gamma_{FS} & \gamma_{FF}-\lambda \end{vmatrix} \tag{9}$$

where

$$\gamma_{SS} = -\eta/2 - \frac{3}{2}\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right)g^{*2} + g^{*4}\left(\frac{\Gamma(D/2)}{2}\right)^2 \Gamma(7-D)\left(\tfrac{65}{3}A + \tfrac{25}{6}B + 14C\right)$$
$$-\left(\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{8-D}{2}\right)^2 \frac{g^{*4}}{16} + 2(2-D)\right),$$

$$\gamma_{SF}=\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{8-D}{2}\Big)g^{*2}-g^{*4}\Big(\frac{\Gamma(D/2)}{2}\Big)^2\Gamma(7-D)(18A+5B+14C)$$
$$+\Big(\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{8-D}{2}\Big)\Big)^2 g^{*4}\frac{1}{24},$$

$$\gamma_{FS}=\frac{3}{2}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{8-D}{2}\Big)g^{*2}-g^{*4}\Big(\frac{\Gamma(D/2)}{2}\Big)^2\Gamma(7-D)(27A+\tfrac{33}{6}B+21C)$$
$$+g^{*4}\Big(\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{8-D}{2}\Big)\Big)^2\frac{1}{16},$$

$$\gamma_{FF}=-\frac{\eta}{2}-3\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{8-D}{2}\Big)g^{*2}+g^{*4}\Big(\frac{\Gamma(D/2)}{2}\Big)^2\Gamma(7-D)(50A+9B+39C)$$
$$-\frac{g^{*4}}{8}\Big(\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{8-D}{2}\Big)\Big)^2+2(D-2).$$

The values of g^* that we have used are taken from Fucito and Marinari (1981) and are obtained by the pseudo ε expansion method (Le Guillou and Zinn-Justin 1980).

Before analysing the results we have to fix the irrelevance condition for the φ^4 operators.

Solving the Callan–Symanzik equation at the fixed point, we obtain that the four-point vertex (a) without and (b) with insertion of the operators is:

$$\begin{aligned}&\text{(a)}\qquad \Gamma^{(4)}_{(\mathrm{R})}(q_i,g^*,m)=m^{2\eta}G(q_i),\\ &\text{(b)}\qquad \Gamma^{(4)}_{k(\mathrm{R})}(q_i,p,g^*,m)=m^{2\eta+\lambda_k}H(q_i,p),\end{aligned}\tag{10}$$

where p is the momentum flowing through the insertion and q_i is that flowing through the external legs. If μ is the scale of the momentum q_i, dimensional analysis tells us that $\Gamma^{(4)}_{(\mathrm{R})}$ and $\Gamma^{(4)}_{k(\mathrm{R})}$ scale like

$$\begin{aligned}&\Gamma^{(4)}_{(\mathrm{R})}(q_i,g^*,m)=m^{2\eta}\mu^{-2+\varepsilon-2\eta}g(q_i/\mu),\\ &\Gamma^{(4)}_{(\mathrm{R})k}(q_i,0,q^*,m)=m^{2\eta+\lambda_k}\mu^{-2\eta-\lambda_k}h(q_i/\mu).\end{aligned}\tag{11}$$

This last vertex function defines also the renormalised coupling constant of O_1 and O_2. The irrelevance condition is that the coupling constants of O_1 and O_2 go to zero faster than g when the scale $\mu\to 0$, and q_i/μ remains finite. In the formula, we have $2-\varepsilon-\lambda_k>0$.

Positive λ_k are a sign that the perturbation induced by φ^4 is not irrelevant. We have now to interpret the fact that at $D=2$ one eigenvalue is positive. All the field-theoretical evaluations of the critical indices of percolation (both the ε expansion and the fixed-dimension method), although they give good results for dimension $5\leqslant D\leqslant 3$, fail in predicting the sign of the exponent η in $D=2$. In fact, just in two dimensions, scaling laws give $\beta=\eta$ (β is the critical index of magnetic susceptibility) and so, going from $D=3$ to $D=2$, the anomalous field dimension η has to change sign, passing from negative to positive values. The positive value of λ_k for $D=2$ points out that the pure φ^3 fixed point is no longer stable and that if we want to have a correct value of η, at the order of two loops, we should probably do a three-coupling-constant expansion.

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